# Exact relativistic solution of disordered radiation with planar symmetry



1977 J. Phys. A: Math. Gen. 10 1679

(http://iopscience.iop.org/0305-4470/10/10/005)

J. Phys. A: Math. Gen., Vol. 10, No. 10, 1977. Printed in Great Britain. © 1977

# Exact relativistic solution of disordered radiation with planar symmetry

A F da F Teixeira†, Idel Wolk† and M M Som‡

† Centro Brasileiro de Pesquisas Físicas, ZC-82 Rio de Janeiro, Brazil
‡ Instituto de Física UFRJ, Cidade Universitária, Rio de Janeiro, Brazil

Received 26 April 1977, in final form 4 July 1977

**Abstract.** An exact solution of the Einstein equations corresponding to an equilibrium distribution of disordered electromagnetic radiation with planar symmetry is obtained. This equilibrium is due solely to the gravitational and pressure effects inherent to the radiation. The distribution of radiation is found to be maximum and finite at the plane of symmetry, and to decrease monotonically in directions normal to this plane.

The solution tends asymptotically to the static plane symmetric vacuum solution obtained by Levi-Civita. Time-like and null geodesics are discussed.

## 1. Introduction

In general relativity there are few exact solutions representing the vacuum fields. The proper interpretations of these solutions depend a great deal on the structure of the sources of the field. There are different structures that can be associated with the source for a particular vacuum field solution.

It is quite worthwhile to explore the possibility of the highly relativistic fluid as the source of plane symmetric vacuum solution first given by Levi-Civita (1918). As is well known gravitation can never produce hydrostatic equilibrium in a finite highly relativistic fluid with $p = \rho/3$, the only possible distribution that can exist is unbounded plane symmetric, asymptotically tending to the plane symmetric vacuum solution.

Tolman (1934, § 109) seems to have been the first to provide sufficient mathematical apparatus for studying that possibility; he explained the circumstances under which radiation may be treated as a special case of a perfect fluid. Klein (1948) first applied Tolman's results to a cosmological situation; he studied a spherically symmetric distribution of disordered electromagnetic radiation in equilibrium. He was able to find only an approximate solution, which he presented as a set of series expansions in terms of his dimensionless radial variable $\kappa p_0 r^2$.

In the present paper we obtained the exact solution of an unbounded plane symmetric distribution of disordered radiation in equilibrium. Similarly to Klein's sphere, our distribution shows a larger condensation in the central regions and dilutes monotonically to a vanishing distribution outwards. In the asymptotic regions our solution goes to Levi-Civita's (1918) plane symmetric vacuum solution. Some time-like and null geodesics are discussed.

## 2. General equations

We start with the static and plane symmetric line element:

$$ds^2 = e^{2\alpha}(dx^0)^2 - e^{2\beta}\,dx^2 - e^{\beta-\alpha}(dy^2 + dz^2), \tag{1}$$

where $\alpha$ and $\beta$ are functions of $x$ alone. The surviving components of the Ricci tensor are:

$$R^0_0 = -e^{-2\beta}\alpha_{11}, \tag{2}$$

$$R^1_1 = -[\beta_{11} + \tfrac{1}{2}(3\alpha_1 + \beta_1)(\alpha_1 - \beta_1)]\,e^{-2\beta}, \tag{3}$$

$$R^2_2 = R^3_3 = \tfrac{1}{2}(\alpha_{11} - \beta_{11})\,e^{-2\beta}, \tag{4}$$

where the subscript 1 means $d/dx$.

Perfect fluids are systems with energy–momentum tensor given by:

$$T^\mu_\nu = (\rho + p)u^\mu u_\nu - p\delta^\mu_\nu, \tag{5}$$

where $\rho$, $p$ and $u^\mu$ are the rest energy density, the pressure and the macroscopic velocity field of the fluid; this last quantity must satisfy $u^\nu u_\nu = 1$. We consider a perfect fluid with equation of state:

$$\rho = 3p \tag{6}$$

(Tolman 1934, § 109); for such a fluid with planar symmetry and under the static condition ($u^1 = u^2 = u^3 = 0$) we have

$$T^\mu_\nu = p(x)\,\mathrm{diag}(3, -1, -1, -1). \tag{7}$$

The Einstein equations:

$$R^\mu_\nu = -\kappa(T^\mu_\nu - \tfrac{1}{2}\delta^\mu_\nu T), \qquad \kappa = 8\pi G/c^4, \tag{8}$$

reduce to the three equations:

$$e^{-2\beta}\alpha_{11} = 3\kappa p, \tag{9}$$

$$[\beta_{11} + \tfrac{1}{2}(3\alpha_1 + \beta_1)(\alpha_1 - \beta_1)]\,e^{-2\beta} = -\kappa p, \tag{10}$$

$$(\alpha_{11} - \beta_{11})\,e^{-2\beta} = 2\kappa p, \tag{11}$$

and the contracted Bianchi identity gives the relation:

$$p_1 + 4p\alpha_1 = 0. \tag{12}$$

## 3. Solution of equations

From (9) and (11) we easily obtain $3\beta = a - bx + \alpha$, where $a$ and $b$ are constants of integration. Since we seek a solution which is regular at the plane of symmetry, we put $g_{00} = -g_{xx} = -g_{yy} = 1$ on the plane $x = 0$; from (1) one finds then that $\alpha(0) = \beta(0) = 0$. So, with the constant $a = 0$, we have

$$3\beta = \alpha - bx, \qquad b = \text{constant}. \tag{13}$$

From (9), (10) and (13) we obtain the equation

$$12\alpha_{11} + (10\alpha_1 - b)(2\alpha_1 + b) = 0, \tag{14}$$

whose solution is:

$$10\alpha = -5bx + 6 \ln (c + d\, e^{bx}), \tag{15}$$

with $c$ and $d$ constants of integration. We also require that our system present mirror symmetry with respect to the plane $x = 0$; we then impose, as another boundary condition, that the normal derivative of the metric coefficient $g_{00}$ be zero on the plane $x = 0$. So, with $\alpha(0) = \alpha_1(0) = 0$, we obtain from (15)

$$c = 1/6, \qquad d = 5/6. \tag{16}$$

The pressure $p$ can now be easily obtained from (9): the result satisfies the relation (12) and is

$$p = (b^2/36\kappa)\, e^{-4\alpha}. \tag{17}$$

If we call $b^2/36\kappa = p_0$ our results become:

$$ds^2 = f^3 e^{-\xi}(dx^0)^2 - f\, e^{-\xi}\, dx^2 - f^{-1}(dy^2 + dz^2), \tag{18}$$

$$p = p_0 f^{-6} e^{2\xi}, \tag{19}$$

where:

$$f(\xi) = [\tfrac{1}{6}(1 + 5\, e^{\xi})]^{2/5}, \qquad \xi(x) = 6(\kappa p_0 x^2)^{1/2} \geqslant 0. \tag{20}$$

In regions close to the central plane ($x = 0$) we have the approximate values:

$$g_{00} = 1 + \xi^2/12, \qquad p = p_0(1 - \xi^2/6), \qquad \xi \ll 1; \tag{21}$$

these results will be used in connection with some special geodesics in the next section.

For studying the properties of the system in regions far from the central plane $x = 0$ we introduce the primed coordinates given by:

$$x'^0/x^0 = (5/6)^{3/5}, \qquad x'/x = y/y' = z/z' = (5/6)^{1/5}, \tag{22}$$

in terms of which the exact solution becomes:

$$ds^2 = h^3 e^{-\eta}(dx^{0'})^2 - h\, e^{-\eta}\, dx'^2 - h^{-1}(dy'^2 + dz'^2), \tag{23}$$

$$p = qh^{-6} e^{2\eta}, \tag{24}$$

where:

$$h(\eta) = (e^{\eta} + 1/5)^{2/5}, \qquad \eta(x') = 5(\kappa q x'^2)^{1/2} \geqslant 0, \qquad q = p_0(6/5)^{12/5}. \tag{25}$$

Then in regions far from the plane $x' = 0$ we have the approximate (asymptotic) solution:

$$ds^2 = e^{\eta/5}(dx^{0'})^2 - e^{-3\eta/5}\, dx'^2 - e^{-2\eta/5}(dy'^2 + dz'^2), \tag{26}$$

$$p = q\, e^{-2\eta/5}, \qquad \eta \gg 1;$$

these results will be discussed later, in connection with the exact Levi-Civita plane symmetric static vacuum solution.

Before closing this section we evaluate the energy content of our system, per unit area on the plane $x' = 0$. We start from the expression of the energy content of a volume element (Tolman 1934, § 92):

$$d^3E = (-g)^{1/2}(2T_0^0 - T)\, dx'\, dy'\, dz', \qquad g = \det g_{\mu\nu}; \tag{27}$$

for our fluid, with line element (23) and pressure (24), we have

$$d^3E/dy'\,dz' = 6qh^{-5}\,e^{\eta}\,dx'. \tag{28}$$

Integrating this differential $d\epsilon'$ of the surface energy density between two planes $x' = \pm$constant we get:

$$\epsilon'(\eta) = (4q/\kappa)^{1/2}(e^{\eta}-1)(e^{\eta}+1/5)^{-1}; \tag{29}$$

for $|x'|$, $\eta \to \infty$ we obtain for the surface energy density a finite value:

$$\epsilon' = (4q/\kappa)^{1/2}. \tag{30}$$

This result will also be discussed later, in connection with the Levi-Civita solution.

## 4. Time-like geodesics

It is well known that if $u^{\mu}$ is the four-velocity of a test particle in geodetic motion and $k^{\mu}$ is a Killing vector field, then $u_{\mu}k^{\mu} = $ constant along the geodetic motion. Since the symmetries of our system are given by:

$$k^{\mu}_{(a)} = \delta^{\mu}_{a}, \qquad a = 0, 2, 3, \tag{31}$$

the possible time-like geodesics have necessarily:

$$u_0 = D^2, \qquad u_2 = -B, \qquad u_3 = -C, \tag{32}$$

where $D^2$, $B$ and $C$ are constants. The contravariant components are

$$u^0 = D^2\,e^{-2\alpha}, \qquad u^2 = B\,e^{\alpha-\beta}, \qquad u^3 = C^{\,\alpha-\beta}, \tag{33}$$

$$(u^1)^2 = [D^4\,e^{-2\alpha} - 1 - (B^2+C^2)\,e^{\alpha-\beta}]e^{-2\beta}, \tag{34}$$

where we used $u^{\nu}u_{\nu} = 1$ to obtain (34).

In view of the difficulty in obtaining subsequent integrals of (33) and (34) with our line element (18) we only consider motions of test particles with velocities small in comparison with that of light, and in regions not far from the central plane $x = 0$. In other words, we take the velocity parameters $B^2$, $C^2$, $D^2 - 1$ and the distance variable $\xi$ all very small. We then obtain from (33) and (34) with the line element (18):

$$dx^0/ds \simeq (1 - 3\kappa p_0 x^2)D^2 \simeq 1, \qquad dy/ds \simeq B, \qquad dz/ds \simeq C, \tag{35}$$

$$(dx/ds)^2 \simeq D^4 - 1 - B^2 - C^2 - 3\kappa p_0 x^2. \tag{36}$$

These equations can now be integrated easily; we call $x^0 = ct$ and obtain the approximate (non-relativistic) time-like geodesics:

$$dx/dt \simeq cA\,\sin\omega t, \qquad dy/dt \simeq cB, \qquad dz/dt \simeq cC, \tag{37}$$

where:

$$A^2 = D^4 - 1 - B^2 - C^2, \qquad \omega = (3\kappa c^2 p_0)^{1/2}. \tag{38}$$

These geodesics represent sinusoidal motions on planes normal to the plane $x = 0$, and with nodes on the plane $x = 0$.

## 5. Null geodesics

Null geodesics are formally obtained from (33), but now $u^\nu u_\nu = 0$. A first integral is then

$$\mathrm{d}y/\mathrm{d}x^0 = B\,\mathrm{e}^{3\alpha-\beta}, \qquad \mathrm{d}z/\mathrm{d}x^0 = C\,\mathrm{e}^{3\alpha-\beta}, \tag{39}$$

$$(\mathrm{d}x/\mathrm{d}x^0)^2 = \mathrm{e}^{2\alpha-2\beta} - (B^2+C^2)\,\mathrm{e}^{5\alpha-3\beta}; \tag{40}$$

the two constants $B$ and $C$ are related to a given initial direction of the null geodesic.

Let us consider a light ray travelling in the plane $z = 0$. Making the constant $C = 0$ in (39) and (40) we obtain for the trajectory in the $(x, y)$ plane the equation:

$$(\mathrm{d}y/\mathrm{d}x)^2 = B^2 f^6\,\mathrm{e}^{-2\xi}(1 - B^2 f^4\,\mathrm{e}^{-\xi})^{-1}, \tag{41}$$

where we used the line element (18). One finds that:

$$B^2 = \sin^2 \nu, \tag{42}$$

$\nu$ being the angle of incidence of the ray on the plane $x = 0$, where $\xi = 0$ and $f = 1$. After crossing this central plane the ray travels outwards until it reaches a maximum distance from the plane $x = 0$; this distance is given by $\mathrm{d}x/\mathrm{d}y = 0$, or

$$\mathrm{e}^{\xi/2} f^{-2} = \sin \nu. \tag{43}$$

After having reached this distance the ray proceeds inwards with identical characteristics. For large angles of incidence on the plane $x = 0$ ($\nu \simeq \pi/2$, $B^2 \simeq 1$) one finds from (43) that the maximum distance reached by the ray is given by:

$$\xi_{\max} \simeq 3(\pi/2 - \nu)^2 \ll 1; \tag{44}$$

for almost normal incidences ($\nu \simeq 0$, $B \simeq 0$) we obtain:

$$\xi_{\max} \simeq -(10/3) \ln \nu \gg 1, \tag{45}$$

while for an incidence of 45° we get:

$$\xi_{\max} \simeq 1{\cdot}5. \tag{46}$$

## 6. Discussion

We obtained the exact unbounded solution given by general relativity to a class ($\rho = 3p$) of perfect fluids with planar symmetry, under static conditions; two physical examples of such fluids are disordered distributions of electromagnetic radiation (Tolman 1934, § 109), and disordered distributions of neutrinos (Klein 1948). Also distributions of colliding particles with randomly oriented ultrarelativistic velocities can be described, in the first approximation, in terms of that class of fluids (Klein 1948).

We found that the gravitational attraction associated with the energy density of these fluids is strong enough to compensate for the repulsion caused by the corresponding pressure. It is then possible to have an isotropic electromagnetic radiation bound together in a static equilibrium configuration solely due to its own gravitation.

We have defined (20) our 'working $x$ variable' $\xi$ such that $\xi(x) = \xi(-x)$; this ensured the mirror symmetry of the system across the plane $x = 0$, since the pressure and all metric coefficients have been expressed in terms of $\xi$. The same remark holds for $\eta(x')$.

The density of our plane symmetric fluid is maximum and finite on the central plane $x=0$, and decreases monotonically to zero in both directions normal to this plane.

In our system the scalar curvature $R^{\mu}_{\mu}$ vanishes everywhere, however $R^{\mu}_{\nu}R^{\nu}_{\mu}=12\kappa^2p^2$ as can easily be obtained from (7) and (8); this quantity is also finite in the central plane and decreases monotonically to zero outwards.

In the central zone ($\xi \ll 1$) the density of the fluid is nearly uniform, as can be seen from (21); as a consequence we obtained a sinusoidal motion (37) for slowly moving test particles in that zone.

It is known from non-relativistic mechanics that an infinite slab of homogeneous fluid of mass density $\mu$ produces internal motions of test particles which are sinusoidal with frequency $\omega^2=\kappa c^4\mu/2$. If we compare this result with ours, $\omega^2=\kappa c^2\rho$, obtained in (38), we find that our fluid with $T^0_0=\rho=3p$ produces a gravitational field which, in the first approximation, resembles that produced by a homogeneous fluid with active mass density $\mu=2\rho/c^2$. This is in agreement with the general result (Tolman 1934, § 110) that active gravitational mass density is $(\rho+3p)/c^2$.

It is also known from non-relativistic gravistatics that an infinite homogeneous slab with surface mass density $\sigma$ produces an external acceleration field which is uniform and directed inwards, of strength $2\pi G\sigma$ (or $\kappa c^4\sigma/4$); to this acceleration field corresponds a Newtonian potential:

$$\phi(x')=(\kappa c^4\sigma/4)|x'|. \tag{47}$$

Newtonian potential $\phi$ is often related to the metric coefficient $g_{00}$ of the relativistic description according to $g_{00}\simeq\exp(2\phi/c^2)$. Indeed, the exact Levi-Civita (1918) static vacuum solution with planar symmetry can be written as:

$$ds^2=e^{2\phi/c^2}(dx^{0'})^2-e^{-6\phi/c^2}\,dx'^2-e^{-4\phi/c^2}(dy'^2+dz'^2), \tag{48}$$

with $\phi$ as given in (47). One finds that this line element coincides with our asymptotic line element (26). Our surface energy density, $\epsilon'=2(q/\kappa)^{1/2}$, obtained in (30) coincides with Levi-Civita's surface density of energy, $c^2\sigma=2(q/\kappa)^{1/2}$, obtained by comparing (25) and (26) with (47) and (48).

Again, in the non-relativistic gravistatics of usual perfect fluids, one finds that the condition for local equilibrium is grad $p=-\mu'$ grad $\phi$, where $p$ is the pressure, $\mu'\gg p/c^2$ is the mass density and $\phi$ is the Newtonian potential. If we now compare this equation with the result $p_1=-(4\rho/3)\alpha_1$ stated in (12) we find that, in a first approximation ($\phi=c^2\alpha$), our fluid behaves with a passive mass density $\mu'=4\rho/3c^2$. For usual incompressible fluids the contribution $p/c^2$ of the pressure $p$ to the passive mass density $\mu'=(\rho+p)/c^2$ (Tolman 1934 § 95) is negligible ($p\ll\rho$), but in our fluid this contribution is considerable and amounts to $p=\rho/3$.

From (29) and (30) one finds that one half of the total energy of our fluid slab is contained between the pair of planes given by $(e^{\eta}-1)(e^{\eta}+1/5)^{-1}=1/2$, or $\eta=\xi=0{\cdot}8$. We may then take the value $2\xi=1{\cdot}6$ as a measure of the effective thickness of our slab. Referring then to (20) one finds that this thickness is given in light years by $|2x|=186p_0^{-1/2}$, with the central pressure $p_0$ in atmospheres; the thickness of the slab is thus seen to decrease with increasing central pressure. Some representative values of radiation pressures are 0·002 atm and $10^7$ atm, corresponding to the situation on a hot star surface and at the beginning of a thermonuclear reaction, respectively (Band 1955). The values of the effective thickness $|2x|$ corresponding to these values for the central pressure $p_0$ are 4000 light years and 20 light days, respectively.

## Acknowledgments

We acknowledge CNPq (Brazil) for the grants 01.120.306 and 01.120.721, and partial support from FINEP (Brazil). Thanks are due to the referees, whose comments helped improve this work.

## References

Band W 1955 *An Introduction to Quantum Statistics* (New York: van Nostrand) p 145
Klein O 1948 *Ark. Mat. Astr. Fys.* A **34** no. 19
Levi-Civita T 1918 *Atti Accad. Naz. Lincei Rc.* **27** 240
Tolman R C 1934 *Relativity, Thermodynamics and Cosmology* (London: Oxford University Press)